MEASUREMENT OF CRACK DISTRIBUTION IN A ROCK
MASS FROM OBSERVATION OF ITS SURFACES

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ABSTRACT
The anisotropy of the crack distribution in a rock mass is characterized by what is termed
the "fabric tensor," and its geometrical interpretation is given. A practical procedure is
presented to determine the internal crack distribution by observing cross-sections of cracks
that appear on plane surfaces of the material by means of the stereological principle. Only
three types of surfaces are necessary for observation.

Key words: anisotropy, crack, fabric tensor, measurement, rock mass, stereology (IGC:
F 0/F 3)

INTRODUCTION
Mechanical properties of a rock is greatly
affected by the distribution of internal cracks.
The crack distribution has two character-
istics. One is the density of the cracks, say
the total crack area per unit volume, and
the other is its anisotropy, i.e., the degree
to which the distribution is deviated from
isotropic distribution. These characteristics
must be expressed by particular quantities,
if we want to incorporate them into constitu-
tive equations which describe mechanical
properties like fracture strength and macro-
scopical elastic moduli. Apparently, the density is described by a scalar quantity, while
the anisotropy is described by a tensor quanti-
and Kanatani (1984a, 1984b) called the
et al (1984) also investigated the correlation
between the fabric tensor and mechanical
properties like fracture strength and macro-
scopic elastic moduli.

When we apply this type of constitutive
equations to real rocks, we must first know
the internal crack distribution, which is very
difficult to measure in practice. What can
be observed is almost always restricted to
the material surface. Hence, we must esti-
mate the internal crack distribution from
the data observed on the surface. This sort
of study, i.e., estimating the three dimen-
sional structure by two dimensional observa-
tions, has been studied as "integral geometry" in mathematics and known as "stereo-
logy" to people in metallurgy, biology and
medicine (Santalo, 1953, 1976; Kendall and
Moran, 1963; Elias, 1967; DeHoff and
Rhines, 1968; Underwood, 1970; Miles and
Serra, 1979; Weibel, 1979, 1980). In particu-
lar, estimation of the size distribution of

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particles inside a material from the size distribution of their cross-sections on the surface has been well studied (e.g., Goldsmith, 1967; Nicholson, 1970; Exner, 1972; Coleman, 1982, 1983; Cruz-Orive, 1983). The same principle applies to estimate the size distribution of internal cracks from the size distribution of their cross-sections on the surface (e.g., see Oda, 1983). Various computational schemes for it and numerical errors involved in them are also studied by Kanatani and Ishikawa (1985).

The estimation of anisotropy from observations on the surface has been also studied, dating back to “Buffon’s needle problem” (Buffon, 1777). The problem was formulated as estimation of the “distribution density” by Hilliard (1967) and as estimation of the “fabric tensor” by Kanatani (1984 b). The same principle is used in computer vision and image processing (Kanatani, 1984 c, 1985 a). If we want to apply these results directly, we must cut the material by a large number of cutting planes and observe the resulting surfaces, which is impossible if the number of material samples are limited. However, if the anisotropy is known to be “weak,” estimation schemes requiring only three types of cross-sections are available (Kanatani, 1985 b). In this paper, we present such a scheme of estimating the crack distribution based on the general theory of Kanatani (1985 b).

**DISTRIBUTION DENSITY AND FABRIC TENSORS**

In the following, we regard a crack as a plane surface without thickness. A non-planar crack is treated as an assembly of several cracks, each being planar. Attach a unit normal vector \( \mathbf{n} \) to each surface. Since there are two possibilities of opposite directions, choose one randomly with probability 1/2. (This is equivalent to dividing a crack area equally into two and attaching to them two normals of opposite directions.) Define the “distribution density” \( f(\mathbf{n}) \) in such a way that \( f(\mathbf{n})d\Omega(\mathbf{n}) \) is the summed area, per unit volume of the region under consideration, of those crack surfaces having normals inside the differential solid angle \( d\Omega \) along \( \mathbf{n} \). (The definition is the same as that of the distribution density of interparticle contacts of a granular material (Kanatani, 1981, 1985).

If the orientation of \( \mathbf{n} \) is described by its spherical coordinates \( \theta \) and \( \phi \) with respect to a Cartesian coordinate system fixed in the material, the distribution density \( f(\mathbf{n}) \) is also regarded as a function of \( f(\theta, \phi) \) of \( \theta \) and \( \phi \), \( 0 \leq \theta < \pi \), \( 0 \leq \phi < 2\pi \) (see Fig.1.). By definition.

\[
\begin{align*}
\mathcal{C} &= \int f(\mathbf{n})d\Omega(\mathbf{n}) \\
&= \int_0^{2\pi} \int_0^\pi f(\theta, \phi)\sin\theta d\theta d\phi
\end{align*}
\]  

(1)

is the “area density”, i.e., the total area of cracks per unit volume. Since \( f(\mathbf{n}) \) or \( f(\theta, \phi) \) is regarded as a function on a unit surface, it can be expressed by the following spherical harmonics expansion (cf. Kanatani, 1984 a).

\[
\begin{align*}
f(\theta, \phi) &= \frac{c}{4\pi} \left[ 1 + \sum_{n=1}^\infty \frac{[1]}{2} a_{nm}\cos \theta \right. \\
&\left. + \sum_{m=1}^n P_n^m(\cos \theta) \left[ a_{nm} \cos m\phi + b_{nm} \sin m\phi \right] \right].
\end{align*}
\]  

(2)

\[
\begin{align*}
a_{nm} &= \frac{2(2n+1)}{c} \frac{(n+m)!}{(n-m)!} \\
&\times \int_0^\pi \int_0^{2\pi} f(\theta, \phi) P_n^m(\cos \theta) \cos m\phi \sin m\phi d\theta d\phi.
\end{align*}
\]  

(3)

Here, \( P_n(z) \) is the \( n \) th Legendre polynomial, \( P_n^m(z) \) is the associated Legendre function,
and \( \sum' \) designates summation with respect to only even indices. Odd terms do not appear because \( f(n) \) or \( f(\theta, \phi) \) is "symmetric" with respect to the origin, i.e., \( f(n) = f(-n) \) or \( f(\theta, \phi) = f(\pi - \theta, \phi + \pi) \). In terms of Cartesian tensors, this expansion is expressed in the following form (cf. Kanatani, 1984a).

\[
f(n) = -\frac{c}{4\pi} [1 + D_{ij} n_i n_j + D_{ijkl} n_i n_j n_k n_l + \cdots].
\]

(4)

Throughout this paper, Einstein's summation convention over repeated indices is adopted. Here, \( D_{ij}, D_{ijkl}, \ldots \) are symmetric deviator tensors (see Kanatani, 1984a) and \( n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). Kanatani (1984a) called \( D_{ijkl} \) the \( n \)th "fabric tensor" of the distribution. It is expressed in terms of the spherical expansion coefficients \( a_{nm} \) and \( b_{nm}, m = 0, 1, \ldots, n \) (Kanatani, 1984a).

In particular, we have for \( n=2 \)

\[
(D_{ij}) = \begin{bmatrix}
-\frac{1}{4} a_{20} + 3a_{22} & 3b_{22} & \frac{3}{2} a_{21} \\
3b_{22} & -\frac{1}{4} a_{20} - 3a_{22} & \frac{3}{2} b_{21} \\
\frac{3}{2} a_{21} & \frac{3}{2} b_{21} & \frac{1}{2} a_{22}
\end{bmatrix}
\]

(5)

In terms of \( f(n) \), Eq. (3) becomes

\[
D_{ijkl} = \frac{2n+1}{2\pi} \frac{2n}{n} N_{(i_1 \cdots i_n)}(n)
\]

(6)

where \( N_{(i_1 \cdots i_n)} \) is the \( n \)th "moment tensor" of \( f(n) \), i.e.,

\[
N_{(i_1 \cdots i_n)} = \int n_{(i_1} \cdots n_{i_n} f(n) d\Omega(n),
\]

(7)

and \( \{ \) designates the "deviator part" of a symmetric tensor (Kanatani, 1984a). In particular, \( N_{ij} \) is related to \( c \) and \( D_{ij} \) by

\[
c = N_{ii},
\]

(8)

\[
D_{ij} = \frac{15}{2c} N_{ij} - \frac{5}{2} \delta_{ij},
\]

(9)

where \( \delta_{ij} \) is the Kronecker delta.

Among the fabric tensors, the most important one is \( D_{ij} \), which is zero for isotropic distribution. Hence, it is a measure of "deviation from isotropy." Those of higher ranks \( D_{ijkl} \), etc. describe higher order fluctuations (Kanatani, 1984a). If higher order fluctuations are neglected, the distribution \( f(n) \) represents a smooth orthogonal anisotropy, having the principal axes of \( D_{ij} \) as the symmetry axes. The distribution takes local maxima or minima along them. The corresponding eigenvalues of \( D_{ij} \) are the ratios of the distribution to the isotropic distribution along these axes. Thus, \( D_{ij} \) has a clear geometrical meaning. On the other hand, Oda (1982, 1983) called the moment tensor of Eq. (7) the "fabric tensor." It is related to ours by Eq. (6) or Eqs. (8) and (9). Eq. (7) can be interpreted as follows. Let \( V \) be the volume of the region in question. Number the cracks in it from 1 through \( N \). Let \( n^{(1)}, \ldots, n^{(N)} \) be their respective unit normals and let \( S^{(1)}, \ldots, S^{(N)} \) be their respective areas. Then,

\[
N_{(i_1 \cdots i_n)} = \frac{1}{V} \sum_{a=1}^{N} n^{(a)}_{i_1} \cdots n^{(a)}_{i_n} S^{(a)}.
\]

(10)

If the crack distribution is nearly isotropic and higher order fluctuations can be neglected, it is sufficient to know the area density \( c \) and the fabric tensor \( D_{ij} \) to estimate the distribution density \( f(n) \). Since \( c \) and \( D_{ij} \) are related to \( N_{ij} \) by Eqs. (8) and (9), it seems that we could compute \( N_{ij} \) by Eq. (10). However, it is very difficult to measure the orientations of the cracks inside a rock. Of course, we can measure the orientations of the cracks near a surface by careful observation, but the process is tedious. In what follows, we give a procedure to estimate \( c \) and \( D_{ij} \) from observations on surfaces by means of the stereological principle. This method would also be applicable to photographs of material surfaces, and hence it could lead to the use of computer image processing techniques in the future.

**PRINCIPLE OF STEREOLOGICAL ESTIMATION**

Imagine the following "stereological measurement." Place a line of orientation \( n \) randomly in a rock material and count the number of intersections with the cracks in
it. Let \( N(n) \), or \( N(\theta, \phi) \) in terms of polar coordinates, be the observed data, namely the number of intersections per unit length of the probe line. Like the distribution density \( f(n) \) or \( f(\theta, \phi) \), it can be expressed as the following spherical harmonics expansion.

\[
N(\theta, \phi) = \frac{C}{4\pi} \left[ 1 + \sum_{n=1}^{\infty} \frac{(\frac{1}{2})}{n} A_{nm} \cos n\phi + B_{nm} \sin n\phi \right],
\]

\[
C = \int_0^{2\pi} \int_0^\pi N(\theta, \phi) \sin \theta \, d\theta \, d\phi,
\]

\[
\begin{bmatrix}
A_{nm} \\
B_{nm}
\end{bmatrix} = \frac{2(2n+1)(n+m)!}{C (n-m)!} \int_0^{2\pi} \int_0^\pi N(\theta, \phi) \sin \theta \cos m\phi \sin m\phi \, d\theta \, d\phi.
\]

Odd terms do not appear because of the "symmetry" with respect to the origin, i.e., \( N(n) = N(-n) \) or \( N(\theta, \phi) = N(\pi - \theta, \phi + \pi) \). The tensor form is

\[
N(n) = \frac{C}{4\pi} \left[ 1 + F_{11}n_1n_1 + F_{12}n_1n_2 + \cdots \right],
\]

and \( F_{11}, \ldots \) is related to \( N(n) \) in the same manner as \( D_{ij} \) is to \( f(n) \) by Eqs. (6) and (7). In particular, \( F_{ij} \) is related to \( A_{20}, \ldots, B_{22} \) by the same way as \( D_{ij} \) is related to \( a_{20}, \ldots, b_{22} \) by Eq. (5).

Now, if \( N(n) \) or \( N(\theta, \phi) \) are given by Eq. (14) or (11), the distribution density is given as follows (Kanatani, 1984 b).

\[
f(\theta, \phi) = \frac{C}{4\pi} \left[ 1 + \sum_{n=2}^{\infty} \frac{1}{n} A_{nm} \cos n\phi + B_{nm} \sin n\phi \right],
\]

\[
\lambda_n = (-1)^{n/2} (-1)^{n-1} (n-1)(n+1) \left( \frac{n}{n/2} \right).
\]

The tensor form becomes

\[
f(n) = \frac{C}{4\pi} \left[ 1 + 4F_{11}n_1n_1 + 4F_{12}n_1n_2 + \cdots \right].
\]

Hence, once \( C \) and \( F_{ij} \) are computed from the observed data \( N(n) \) or \( N(\theta, \phi) \), the area density \( c \) and the fabric tensor \( D_{ij} \) are obtained by

\[
c = \frac{C}{2\pi}, \quad D_{ij} = 4F_{ij},
\]

A most straightforward way to compute \( C \) and \( F_{ij} \) is to approximate the integrals of Eqs. (12) and (13) by appropriate summations. For details, see Kanatani (1984 b). In order to do so, however, we must cut the material with planes of various orientations. This means we must prepare a large number of material samples and cut them with planes of various orientations. This becomes difficult when the amount of the sample material is limited. Here, we seek ways of estimation requiring only a small number of orientations of the cutting planes. This is possible if high order fluctuations of the distribution \( f(n) \) can be neglected.

**PROCEDURE OF STEREOLOGICAL ESTIMATION**

Suppose high order fluctuations of the distribution density \( f(n) \) can be neglected and \( f(n) \) is described by

\[
f(n) = \frac{c}{4\pi} \left[ 1 + D_{ij}n_in_j \right].
\]

If this is the case, we say that the anisotropy of the crack distribution is "weak." If the anisotropy is weak, the observed data \( N(n) \) becomes by Eq. (15)

\[
N(n) = \frac{C}{4\pi} \left[ 1 + F_{ij}n_in_j \right].
\]

This expression has 6 parameters, 1 for \( C \) and 5 for \( F_{ij} \) (a symmetric deviator tensor). Hence, we need, in principle, only 6 data to determine them. However, since actual measurement always involves random errors, the necessary data must have the form of sums or averages of a large number of observations in order to cancel out possible random errors.

Now, consider the following integrations.

\[
M(z) = \int_{c(z)} N(n) \, ds,
\]

(21)
\[ M_{14}^{(z)} = \int_{C(x)} n_1 n_2 N(\mathbf{n}) \, ds. \]  
(22)

Here, \( C(x) \) is a unit circle on the \( xy \)-plane around the \( z \)-axis, and \( \int_{C(x)} \, ds \) designates the line integral along \( C(x) \) (Fig. 2). If we substitute Eq. (20) in these, noting \( F_{ij} = 0 \), we obtain the following expressions (For a more general case, see Kanatani, 1985b).

\[ M^{(z)} = \frac{C}{2} \left( \frac{1}{2} F_{12} \right), \]  
(23)

\[ M_{12}^{(z)} = \frac{C}{8} F_{12}. \]  
(24)

Consider the same quantities defined on the \( yz \)- and \( zx \)-planes as well. Then, as is easily seen, we can obtain \( C \) and \( F_{ij} \) in terms of them as follows.

\[ C = \frac{2}{3} \left( M^{(z)} + M^{(y)} + M^{(x)} \right), \]  
(25)

\[ F_{11} = \frac{2(-2M^{(z)} + M^{(y)} + M^{(x)})}{(M^{(z)} + M^{(y)} + M^{(x)})}, \]  
(26)

\[ F_{22} = \frac{2(M^{(z)} - 2M^{(y)} + M^{(x)})}{(M^{(z)} + M^{(y)} + M^{(x)})}, \]  
(27)

\[ F_{33} = \frac{2(M^{(z)} + M^{(y)} - 2M^{(x)})}{(M^{(z)} + M^{(y)} + M^{(x)})}, \]  
(28)

\[ F_{14} = \frac{12 M_{14}^{(z)}}{(M^{(z)} + M^{(y)} + M^{(x)})} (= F_{11}), \]  
(29)

\[ F_{23} = \frac{12 M_{23}^{(z)}}{(M^{(z)} + M^{(y)} + M^{(x)})} (= F_{22}), \]  
(30)

\[ F_{31} = \frac{12 M_{31}^{(z)}}{(M^{(z)} + M^{(y)} + M^{(x)})} (= F_{13}). \]  
(31)

These relations can be used for estimation of \( C \) and \( F_{ij} \). Cut the material randomly with a plane parallel to the \( xy \)-plane and draw on it a line making angle \( k\pi/N \), \( k = 0, 1, \ldots, N-1 \) from the \( x \)-axis (Fig. 3).

Let \( N_k^{(z)} \) be the number of intersections with the cracks per unit length of the line. Then, approximate \( M^{(z)} \) and \( M_{ij}^{(z)} \) by

\[ M^{(z)} = 2\pi \sum_{k=0}^{N-1} \frac{N_k^{(z)}}{N}, \]  
(32)

\[ M_{14}^{(z)} = \pi \sum_{k=0}^{N-1} \frac{N_k^{(z)} \sin(2\pi k/N)}{N}. \]  
(33)

Do the same thing for the \( yz \)- and the \( zx \)-plane and compute \( M^{(y)} \), \( M_{12}^{(y)} \), \( M^{(y)} \) and \( M_{31}^{(y)} \). Then, \( C \) and \( F_{ij} \) are given by Eqs. (25)–(31). The area density \( C \) and the fabric tensor \( D_{ij} \) are given by Eqs. (18).

Thus, we need to cut the material with only three different planes and hence to prepare only three material samples. If a rectangular box-shaped sample is available, we need to observe only its three faces. Of course, the accuracy is improved if we cut the material with equidistant parallel planes of each orientation and draw on it parallel lines with the separation equal to that of the parallel planes. Then, the average over the observations is taken, the length of each line being the weight.

CONCLUDING REMARKS

In this paper, we have shown a procedure to estimate the internal crack distribution of a rock mass from observations on its surface planes. An important idea behind this is the concept of the "distribution density" \( f(n) \) or \( f(\theta, \phi) \) which characterizes
the distribution of internal cracks. Also important is the fact that the distribution density \( f(n) \) is described by the “area density” \( c \) and the “fabric tensors” \( D_{ik} \), \( D_{ijk} \), etc. The second rank fabric tensor \( D_{ij} \) is the most important one, because it describes the degree of deviation from isotropic distribution. Since any constitutive equations of rock materials must be tensor equations, it is expected that they incorporate the effects due to the crack distribution in terms of \( c \) and \( D_{ij} \). Our method of stereological estimation directly determines \( c \) and \( D_{ij} \) from observations on surfaces when higher order effects can be neglected. We employed the count of intersections between probe lines and crack cross-sections, and this seems most practical in actual experiments. However, we could have also used the length of crack cross-sections which appear on a plane surface of a given orientation. For details, see Kanatani (1985 b).

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